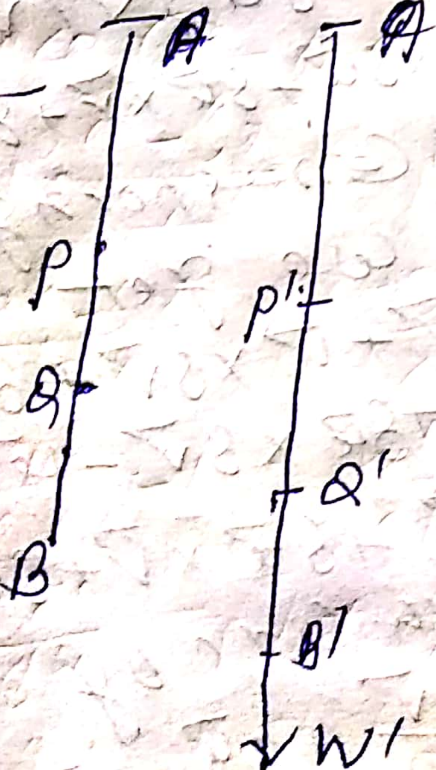
B.Sc (Maths) part-II, paper IV

Elastic springs, strings and HOOK'S Law

Theorem:— A uniform extensible string of weight W and natural lenth L is suspended from a fixed point and at the other end is hung a weight W'. If λ be the Coefficient of elasticity show that the whole extension of the strings is

$$\frac{L}{\lambda}\left[\frac{W}{2}+W'\right]$$

proof:—



let AB = unstretched lenth of string $= L$

$A'B'$ = stretched lenth of string After stretched

$PQ = \delta x$ change to $P'Q' = \delta x'$

$\frac{W}{2}$ is weight of unit lenth of string. Also

weight of the portion P'B'
= weight of the portion PB

$= \frac{W}{2}(l-x)$

For the equilibrium of the element $\delta x'$ we have

Tension in the element P'Q' acting upward = weight W' suspended from the end B' acting downwards + weight of portion P'B' acting downwards —— ①

But by Hook's law

Tension T in P'Q' $= \lambda \frac{\delta x' - \delta x}{\delta x}$

Hence ① becomes

$$\lambda \frac{\delta x' - \delta x}{\delta x} = W' + \frac{W}{2}(l-x)$$

$$\Rightarrow \frac{\delta x' - \delta x}{\delta x} = \frac{W'}{\lambda} + \frac{W}{2\lambda}(l-x)$$

$$\Rightarrow \frac{\delta x'}{\delta x} - 1 = \frac{W'}{\lambda} + \frac{W}{\lambda} - \frac{Wx}{2\lambda}$$

$$\Rightarrow \frac{\delta x'}{\delta x} = 1 + \frac{1}{\lambda}(W + W') - \frac{Wx}{2\lambda}$$

$$\Rightarrow \delta x' = \left[1 + \frac{1}{\lambda}(W + W') - \frac{W}{l\lambda}\right]\delta x$$

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$$\int_0^l dx' = \int_0^l \left[1 + \frac{1}{\lambda}(W + Wl) - \frac{W}{l\lambda}x\right] dx$$

$$\Rightarrow [x']_0^L = \left\{1 + \frac{1}{\lambda}(W + Wl)\right\}[x]_0^l - \frac{W}{2l\lambda}[x^2]_0^l$$

$$\Rightarrow L = \left[1 + \frac{1}{\lambda}(W + Wl)\right]l - \frac{Wl^2}{2l\lambda}$$

$$\Rightarrow L = l + \frac{Wl}{\lambda} + \frac{Wl^2}{\lambda} - \frac{Wl}{2\lambda}$$

$$\Rightarrow L = l + \frac{Wl}{2\lambda} + \frac{Wl^2}{\lambda}$$

$$\Rightarrow L - l = \frac{l}{\lambda}\left[Wl + \frac{W}{2}\right]$$

∴ Extension produced

$$= \frac{l}{\lambda}\left(Wl + \frac{W}{2}\right)$$

Problem ① Find the work done is extending a light elastic string to double its length.

Soln: — Let l be the natural length of the string. If the string stretched to a length x then tension T, by Hook's Law is given by

$T = \lambda \cdot \frac{x-l}{l}$ where λ is modulus of elasticity.

The work done against the tension T in stretching to a further distance δx is $T \cdot \delta x$ i.e $\lambda \cdot \frac{x-l}{l} \delta x$

Hence the total work done in extending it from l to $2l$

$$= \int_{l}^{2l} \lambda \frac{x-l}{l} dx = \frac{\lambda}{l}\left[\frac{1}{2}x^2 - lx\right]_{l}^{2l}$$

$$= \frac{\lambda}{l}\left[\frac{1}{2}(4l^2 - l^2) - l(2l - l)\right]$$

$$= \frac{\lambda}{l}\left[\frac{3}{2}l^2 - l^2\right]$$

$$= \frac{1}{2}\lambda l$$